We start w/ the following lemma.

<u>Lemma</u>:- $\tilde{R}_{max}$ is bounded for the normalized RF on M^3 w/ $Ric(0) > 0$.

<u>Proof</u>:- We have from the pinching estimates that

$$\tilde{Rc} = Rc \geq 2\beta^2 \tilde{R}_{min}\, \tilde{g}$$

for some $\beta > 0$. So Myers's theorem $\Rightarrow$

$$\mathrm{diam}(M)_{\tilde{g}} \leq \frac{\pi}{\beta\sqrt{\tilde{R}_{min}}} .$$

$\because$ the volume of the manifold is constant w.r.t. $\tilde{g}$, say V_0 and $\tilde{Rc} = Rc \geq 0 \Rightarrow$ by Bishop–Günther volume comparison thm, we have

$$V_0 = \mathrm{Vol}\left(B(p, \mathrm{diam}(M))\right) \leq V_3^0(\mathrm{diam}(M))$$
$$= \frac{4\pi}{3}(\mathrm{diam}(M))^3$$

$$\Rightarrow \left(\frac{3V_0}{4\pi}\right)^{\frac{1}{3}} \leq (\mathrm{diam}\, M) \leq \frac{\pi}{\beta\sqrt{\tilde{R}_{min}}}$$

$\Rightarrow$ $\tilde{R}_{min}$ is bounded from above.

Also, recall that $\dfrac{R_{min}}{R_{max}} \geq \dfrac{1}{C}$ but the LHS is scale-invariant $\Rightarrow \dfrac{\tilde{R}_{min}}{\tilde{R}_{max}} \geq \frac{1}{c} \Rightarrow \tilde{R}_{max}$ is also bounded from above.

$\boxed{\checkmark}$

The next theorem proves that the normalized RF exists for all time.

Theorem:- $(M^3, g(t))$ be a sol^n to the RF w/ $Ric(0) > 0$. Then $\tilde{g}(t)$ exists for all time..

proof:- We first prove that $\int_0^T R_{max}(t)\,dt = \infty$.

Note that this is not obvious as there are continuous unbounded function whose integral are finite.

set $f(t) = e^{2\int_0^t R_{max}(u)du} \cdot R_{max}(0)$

$\Rightarrow \quad f'(t) = 2R_{max}(t) f$

$$\Rightarrow \frac{\partial}{\partial t}(R-f) = \Delta R + 2|Rc|^2 - 2R_{max} f$$
$$= \Delta(R-f) + 2|Rc|^2 - 2R_{max} f$$

(b/c f doesn't involve any space variable $\Rightarrow \Delta f = 0$)

$$\leq \Delta(R-f) + 2R_{max}(R-f)$$

(as for $Rc \geq 0$, $|Rc|^2 \leq R^2$)

Now at $t=0$ $\quad R-f = R(0) - 1.\, R_{max}(0) \leq 0.$

The corresponding ODE is $\dfrac{d\varphi}{dt} = 2R_{max}\varphi$

i.e, $\dfrac{d}{dt}\log|\varphi(t)| = 2R_{max}(t)$ $\Rightarrow$ $\varphi(t)$ doesn't change sign

$\Rightarrow$ $R-f \leq 0$ is preserved along RF.

Now $R_{max} \to \infty$ as $t \to T$ $\Rightarrow$ $f \to \infty$ as $t \to T$

$\Rightarrow$
$$\lim_{t\to T} \int_0^t R_{max}(u)\,du = \infty.$$

Having proven that, we look at the corresponding integral for the normalized flow to get

$$\int_0^{\tilde{T}} \tilde{R}_{max}(\tau)\,d\tau = \int_0^T R(t)\,dt = \infty$$

as $\tilde{R}_{max}\,d\tau = (\psi^{-1}R_{max})(\psi\,dt)$

But $\tilde{R}_{max}$ is bounded $\Rightarrow$ the region of integration must be infinite $\Rightarrow$ $\tilde{T} = \infty$.

Convergence of the Normalized Ricci flow

Want to prove that NRF converges as $\tau \to \infty$ to a smooth metric $\tilde{g}_\infty$ of constant positive sectional curvature.

The idea of the proof is similar to the proof of the characterization of the maximal existence time :- we proved that if we have bounds on the curvature then the limit metric exists and is continuous and then using the derivative estimates, we proved uniform bounds on spatial and time derivatives of the metric $\Rightarrow$ we can take limits of higher order derivatives of g and $\Rightarrow$ limit metric is smooth.

So from the same ideas as before, if we want to show that $\tilde{g}(\infty)$ exists and is continuous then we must show that $\exists\ C < \infty$ s.t

$$\int_0^\infty \left|\frac{\partial}{\partial \tau}\tilde{g}\right|_{\tilde{g}} d\tau \leq C.$$

$$\Updownarrow \qquad \text{——} \ (1)$$

$$\int_0^\infty \left|\tilde{R}c - \frac{\tilde{r}}{3}\tilde{g}\right| d\tau < C.$$

Now the pinching estimates tell us that

$$\frac{\left|Rc - \frac{1}{3}Rg\right|^2}{R^2} \leq C R^{-\delta}$$

We'll prove that the integrand in ① is bounded by a decaying exponential, so even if we integrate from 0 to ∞ as in the case of NRF, we'll be fine.

we prove the following for this:-

Lemma 1 If $(M^3, \tilde{g}(\tau))$ is a solution of NRF w/ $Ric(0)>0$ then $\exists\ \epsilon>0$ s.t. $\tilde{R}\geq\epsilon\ \forall\ \tau>0$.

Lemma 2 $\exists$ constants $C, \delta>0$ s.t.

$$|\tilde{E}| = \left|\tilde{Rc}-\frac{\tilde{R}}{3}\tilde{g}\right| \leq C e^{-\delta\tau}.$$

so if we prove that $|\tilde{R}-\tilde{r}|$ is exponentially bounded then we can combine it w/ Lemma 2 to get a bound on the integrand in ①.

Lemma 3 $\exists$ constants $C, \delta>0$ s.t.

$$\tilde{R}_{max}-\tilde{R}_{min} < Ce^{-\delta\tau}.$$

We'll prove the lemmas later, for now let's use them to prove the theorem.

Theorem If $(M^3, \tilde{g}(\tau))$ is a sol^n of NRF w/ $Ric(0)>0$. then $\tilde{g}(\tau)$ exists for all time and converges uniformly as $\tau \to \infty$ to a continuous metric $\tilde{g}(\infty)$.

__Proof.__ From Lemma 2 & 3 above, we get

$$\int_0^\infty \left|\frac{\partial \tilde{g}}{\partial \tau}\right| d\tau = \int_0^\infty \left|\tilde{Rc} - \frac{\tilde{r}}{3}\tilde{g}\right| d\tau$$

$$\leq \int_0^\infty \left|\tilde{Rc} - \frac{\tilde{R}}{3}\tilde{g}\right| + \left|\frac{\tilde{R} - \tilde{r}}{3}\tilde{g}\right| d\tau$$

$$< \int_0^\infty C e^{-\delta\tau} d\tau < \infty$$

$\tilde{R} - \tilde{r} = \tilde{R} - \dfrac{\int \tilde{R}\,\tilde{vol}}{\int \tilde{vol}}$
$\leq \tilde{R}_{max} - \tilde{R}_{min}$
$\leq e^{-\delta\tau}$ as R is bounded.

$\Rightarrow$ as in the characterization of singular time case, $\tilde{g}(\tau)$ converges uniformly to a continuous metric $\tilde{g}(\infty)$ as $\tau \to \infty$.

$\boxed{}$

The next step is to prove that the convergence is C^∞ so that $g(\infty)$ can be smooth and so that the curvature of the normalized flow converge to the corresp- -onding curvatures of the limit metric, as the curvatures are 2^{nd}-order in the metric. We need this so we can conclude that the pinching results we have proven for the flow leads to a similar results for the limit metric and so the limit metric having constant curvature.

Theorem The limit metric $\tilde{g}(\infty)$ is smooth and the convergence of $\tilde{g}(\tau)$ to $\tilde{g}(\infty)$ as $\tau \to \infty$ is uniform in every C^m norm.

assuming this theorem we can state and prove the final theorem for the course.

Theorem:- $\tilde{g}(\infty)$ is a smooth metric w/ constant positive sectional curvature.

Proof:- $\tilde{g}(\tau) \longrightarrow \tilde{g}(\infty)$ in C^0, C^1 and C^2 norms.

$\Rightarrow$ taking the limit implies that the Einstein tensor $\tilde{E}_\infty$ of $\tilde{g}(\infty)$ vanishes as

$$|\tilde{E}_\infty| = \lim_{\tau \to \infty} |\tilde{E}(\tau)| \leq \lim_{\tau \to \infty} C e^{-\delta\tau} = 0$$

$\therefore$ $\tilde{g}(\infty)$ is an Einstein metric $\Rightarrow$ $\tilde{g}(\infty)$ has constant positive sectional curvature.

This also proves the Poincaré conjecture for M^3 which admits a metric of positive Ricci curvature.

Now we'll prove Lemma 1, 2, 3 & the theorem.

Lemma 1 If $(M^3, \tilde{g}(\tau))$ is a solution of NRF w/ $Ric(0) > 0$ then $\exists\ \epsilon > 0$ s.t. $\tilde{R} \geq \epsilon\ \forall\ \tau > 0$.

Proof :- We use the following result to prove this lemma

If M is simply-connected, $\dim \geq 3$ w/ sectional curvature b/w $\frac{1}{4}K$ and K then the inj. radius of $M \geq \frac{\pi}{\sqrt{K}}$.

Now, we proved $\min \nu(x,t) \geq (1-\epsilon) \max \lambda(x,t)$

$\Rightarrow \quad \frac{\nu(x,t)}{\lambda(y,t)} \longrightarrow 1$ as $t \to T$, uniformly for all $x, y \in M$. By scale-invariance, $\frac{\tilde{\nu}}{\tilde{\lambda}} \longrightarrow 1$ uniformly as $T \to \infty$.

$\Rightarrow$ M is becoming as pinched as we want $\Rightarrow$ will be $\frac{1}{4}$-pinched and the constant K in the previous result be equal to a multiple $\tilde{R}_{min}$.

So applying the result above to the universal cover N of M. By the Bishop-Günther vol. comparison thm, we get

$$vol(N) \geq C\, inj(N)^3 \geq C\left(\frac{\pi}{\sqrt{K}}\right)^3 \geq C R_{max}^{-3/2}$$

Also, the Ricci tensor, $Rc \geq 2\beta^2 R g$ $\Rightarrow$ by Myers's thm, $\pi_1(M)$ is finite $\Rightarrow$

$$vol(N) = |\pi_1(M)|\, vol(M) = \text{constant}$$

$\Rightarrow$ $\tilde{R}_{max}$ has a positive lower bound and also $\tilde{R}_{min}$ has a lower bound. $\boxed{4}$

We'll need to use the maximum principle and would like to use the evolution equations for the unnormalized flow. Now along the NRF, $\tilde{g} = \psi g$ and $\therefore$ any tensor $\tilde{P} = \psi^n P$.

<u>Lemma</u> :- If P satisfies $\frac{\partial P}{\partial t} = \Delta P + Q$ then $\tilde{Q} = \psi^{n-1} Q$ and

$$\frac{\partial \tilde{P}}{\partial \tau} = \tilde{\Delta}\tilde{P} + \tilde{Q} + \frac{2\tilde{r}n}{3}\tilde{P}.$$

<u>Proof</u> :- Recall $\frac{\partial \tau}{\partial t} = \psi \Rightarrow \frac{\partial \tilde{P}}{\partial \tau} = \frac{\partial \tilde{P}}{\partial t} \cdot \frac{\partial t}{\partial \tau} = \psi^{-1}(\cdot)$

Also, $\Delta = g^{ij}\nabla_i\nabla_j \Rightarrow \tilde{\Delta} = \psi^{-1}\Delta \Rightarrow \tilde{Q} = \psi^{n-1}Q$.

$$\therefore \frac{\partial \tilde{P}}{\partial \tau} = \psi^{-1}\frac{\partial(\psi^n P)}{\partial t} = \psi^{n-1}\frac{\partial P}{\partial t} + n\psi^{n-2}\frac{d\psi}{dt}P$$

$$= \psi^{n-1}(\Delta P + Q) + n\psi^{n-1}\left(\frac{2\tilde{r}}{3}\psi^{2}\right)P \quad \left(\text{it was } \frac{d\psi}{dt} = \frac{2\tilde{r}}{n}\psi^2\right)$$

$$= \tilde{\Delta}\tilde{P} + \tilde{Q} + \frac{2\tilde{r}n}{3}\tilde{P}.$$

<u>Lemma 2</u> $\exists$ constants $C, \delta > 0$ s.t.

$$|\tilde{E}|, \quad |\tilde{R} - \tilde{R}\tilde{g}| \leq Ce^{-\delta\tau}.$$

$|E| = |R - \frac{R}{3} g| = \ldots$

so if we prove that $|\tilde{R} - \tilde{r}|$ is exponentially bounded then we can combine it w/ Lemma 2 to get a bound on the integrand in ①.

Proof :- We know that $|\tilde{E}|^2 \leq \frac{(\tilde{\lambda} - \nu)^2}{4}$, so we'll prove it for the latter. We'll show

$$\tilde{\lambda} - \tilde{\nu} \leq C e^{-\delta\tau} \underbrace{(\tilde{\mu} + \tilde{\nu})}_{}$$

is bounded as $\tilde{R}_{max} \leq C$.

Using the Uhlenbeck trick for the NRF $\frac{\partial u^i_a}{\partial \tau} = \tilde{R}^i_l u^l_a - \frac{\tilde{r}}{2} u^i_a$

one can check that

$$\frac{\partial}{\partial \tau} \tilde{R}_{abcd} = \tilde{\Delta} \tilde{R}_{abcd} + 2 (\tilde{B}_{abcd} - \tilde{B}_{abdc} + \tilde{B}_{acbd} - \tilde{B}_{adbc}) - \tilde{r} \tilde{R}_{abcd}.$$

So for the eigenvalues

$$\frac{d}{d\tau} \begin{pmatrix} \tilde{\lambda} \\ \tilde{\mu} \\ \tilde{\nu} \end{pmatrix} = \begin{pmatrix} \tilde{\lambda}^2 + \tilde{\mu}\tilde{\nu} - \tilde{r}\tilde{\lambda} \\ \tilde{\mu}^2 + \tilde{\lambda}\tilde{\nu} - \tilde{r}\tilde{\mu} \\ \tilde{\nu}^2 + \tilde{\lambda}\tilde{\mu} - \tilde{r}\tilde{\nu} \end{pmatrix}$$

apply the v.b. max. principle w/ the set

$$K = \{ M \mid e^{\delta\tau} (\tilde{\lambda}(M) - \tilde{\nu}(M)) - C(\tilde{\nu}(M) + \tilde{\mu}(M)) \leq 0 \}$$

for some C and δ to be chosen.

So, we get

$$\frac{d}{dt}\log\left(e^{\delta t}\frac{\tilde{\lambda}-\tilde{\nu}}{\tilde{\mu}+\tilde{\nu}}\right)=\frac{1}{e^{\delta t}\frac{\tilde{\lambda}-\tilde{\nu}}{\tilde{\mu}+\tilde{\nu}}}\cdot\left(\delta\cdot e^{\delta t}\frac{\tilde{\lambda}-\tilde{\nu}}{\tilde{\mu}+\tilde{\nu}}+e^{\delta t}\partial_t(\;)\right)$$

$$\delta-\frac{2\tilde{\mu}^2}{\tilde{\mu}+\tilde{\nu}} \qquad = \delta-(\tilde{\mu}-\tilde{\nu})-\frac{\tilde{\mu}^2+\tilde{\nu}^2}{\tilde{\mu}+\tilde{\nu}}$$

$$\boxed{\begin{aligned}
&\frac{\mu+\nu}{\lambda-\nu}\left(\frac{(\partial_t\lambda-\partial_t\nu)(\mu+\nu)-(\partial_t\mu+\partial_t\nu)(\lambda-\nu)}{(\mu+\nu)^2}\right)\\
&=\frac{1}{(\lambda-\nu)(\mu+\nu)}\Big\{(\lambda^2+\mu\nu-r\lambda-\nu^2-\lambda\mu+r\nu)(\mu+\nu)\\
&\qquad-(\mu^2+\lambda\nu-r\mu+\nu^2+\lambda\mu-r\nu)(\lambda-\nu)\Big\}\\
&=\frac{1}{(\lambda-\nu)(\mu+\nu)}\Big\{\lambda^2\mu+\lambda^2\nu+\mu^2\nu+\mu\nu^2-r\lambda\mu-r\lambda\nu-\nu^2\mu-\nu^3\\
&\qquad-\lambda\mu^2-\lambda\mu\nu+r\nu\mu+r\nu^2\\
&\qquad-\lambda\mu^2+\nu\mu^2-\lambda^2\nu+\lambda\nu^2+r\mu\lambda-r\mu\nu-\nu^2\lambda\\
&\qquad+\nu^3-\lambda^2\mu+\lambda\mu\nu+r\nu\lambda-r\nu^2\Big\}\\
&=\frac{\mu^2(\nu-\lambda)+\mu^2(\nu-\lambda)}{(\lambda-\nu)(\mu+\nu)}=\frac{-2\mu^2}{(\mu+\nu)}
\end{aligned}}$$

$$\leq \delta-\frac{1}{2}(\tilde{\mu}+\tilde{\nu}).$$

Now from Lemma 1, choose $\epsilon>0$ s.t.

$$2\epsilon\leq\tilde{\lambda}+\tilde{\nu}+\tilde{\mu}\leq(1+B)(\tilde{\mu}+\tilde{\nu}) \text{ for some}$$

$B > 0$.

as $\dfrac{\tilde{\lambda}}{\tilde{\mu}+2\tilde{\nu}} \leq \dfrac{\lambda}{\mu+2\nu} \leq B$ from the pinching estimates.

Choose δ small enough so that

$$\delta - \frac{\epsilon}{1+B} < 0$$

$\therefore$

$$\frac{d}{dt} \log\left(e^{\delta\tau} \frac{\tilde{\lambda} - \tilde{\nu}}{\tilde{\mu}+\tilde{\nu}} \right) \leq 0$$

$\therefore$ K is preserved by the ODE and the result is proved.

$\boxtimes$

<u>Lemma 3</u> $\exists$ constants $C, \delta > 0$ s.t.

$$\tilde{R}_{max} - \tilde{R}_{min} < Ce^{-\delta\tau}.$$

<u>Proof.</u> Same idea as before, to have an exponential bound on $|\tilde{\nabla}\tilde{R}|$ and then integrate along paths and having an uniform upper bound on the diameter will give the result.

We do a similar max. principle argument to

$$G = \frac{|\nabla R|^2}{R^2} + \alpha |E|^2, \quad \alpha > 0 \text{ chosen later.}$$

Now $\tilde{G} = \varphi^{-2} G$, so we can show that

$$\frac{\partial G}{\partial t} \leq \Delta G + \beta R|E|^2 \Rightarrow$$

$$\frac{\partial \tilde{G}}{\partial \tau} \leq \tilde{\Delta} \tilde{G} + \beta \tilde{R} |\tilde{E}|^2 - \frac{4\tilde{r}}{3} \tilde{G}$$

$$\leq \tilde{\Delta} \tilde{G} + C e^{-\delta \tau} - \delta \tilde{G}$$

$$\Rightarrow \quad \frac{\partial}{\partial \tau} \left(e^{\delta \tau} \tilde{G} - C_\tau \right) \leq \tilde{\Delta} \left(e^{\delta \tau} \tilde{G} - C_\tau \right)$$

$\Rightarrow \quad e^{\delta \tau} \tilde{G} - C_\tau \leq C \Rightarrow \tilde{G}$ is exponentially decaying. $\Rightarrow \quad |\tilde{\nabla} \tilde{R}|^2$ is also decaying exponentially as $\tilde{R}$ is bounded above.

$\boxtimes$

Now we can prove the final theorem.

<u>Theorem</u> The limit metric $\tilde{g}(\infty)$ is smooth and the convergence of $\tilde{g}(\tau)$ to $\tilde{g}(\infty)$ as $t \to \infty$ is uniform in every C^m norm.

<u>Proof</u>:– Recall from the proof the max. existence criterion thm that we need to prove

$$\int_0^\infty |\partial_\tau \partial^k \tilde{g}| \, d\tau < \infty$$

which will be proved if we can show that

$$|\partial^k F| \leq C e^{-\delta\tau}$$

w/ $F = \tilde{Rc} - \frac{1}{3}\tilde{r}\tilde{g}$.

We'll prove that the derivatives of F are exponentially bounded.

Enough to show the bounds for $\tilde{Rc}$, i.e,

<u>Lemma</u> :- $\exists$ $C_k, \delta_k > 0$ s.t.

$$|\tilde{\nabla}^k \tilde{Rc}| \leq C_k e^{-\delta_k \tau} \quad \forall\ k \geq 1.$$

<u>Proof.</u> Note that the metric is not becoming Ricci flat as $\tau \to \infty \Rightarrow k=0$ is NOT true. So we prove this by induction by we work from $k \geq 1$.

The idea is similar to the proof of Shi's estimates and the gradient estimates for R.

Note $|\tilde{Rc}| \leq |Rc - \frac{1}{3}\tilde{R}\tilde{g}| + \frac{|\tilde{R}|}{3} \leq C$.

Assume $|\tilde{\nabla}^j \tilde{Rc}| \leq Ce^{-\delta\tau}$ for $1 \leq j \leq k-1$.

$\therefore$ in dim 3, Rm is just a combination of Rc terms

$\Rightarrow$ the evolution for Rm from the proof of Shi's estimates can be used for Rc $\Rightarrow$

$$\frac{\partial}{\partial t}|\nabla^k Rc|^2 = \Delta|\nabla^k Rc|^2 - 2|\nabla^{k+1}Rc|^2 + \sum_{j=0}^{k} \nabla^j Rc * \nabla^{k-j} Rc * \nabla^k Rc.$$

for the NRF, we'll get an extra term of $\left(\frac{2n\tilde{r}}{3}\right)|\nabla^k \tilde{Rc}|^2$

and $\because$ $\tilde{r} \leq CR$ $\Rightarrow$ the above term will be subsumed in the $j=0$ case in the summation.

$\therefore$ Using the Induction hypo:-

$$\partial_t|\tilde{\nabla}^k \tilde{Rc}|^2 \leq \tilde{\Delta}|\tilde{\nabla}^k \tilde{Rc}|^2 - 2|\tilde{\nabla}^{k+1}\tilde{Rc}|^2 + \sum \tilde{\nabla}^j \tilde{Rc} * \tilde{\nabla}^{k-j}\tilde{Rc} * \tilde{\nabla}^k \tilde{Rc}$$

$$\leq \tilde{\Delta}|\tilde{\nabla}^k \tilde{Rc}|^2 - 2|\tilde{\nabla}^{k+1}\tilde{Rc}|^2 + Ce^{-\delta\tau} + \tilde{Rc} * (\tilde{\nabla}^k \tilde{Rc})^{*2}$$

$$\leq \tilde{\Delta}|\tilde{\nabla}^k \tilde{Rc}|^2 - 2|\tilde{\nabla}^{k+1}\tilde{Rc}|^2 + Ce^{-\delta\tau} + B_k|\tilde{\nabla}^k \tilde{Rc}|^2.$$

NOT good enough as we do not get exponential decay.

So we use the same trick as the Shi's estimates but instead of adding multiples of $|\tilde{Rc}|^2$ which DO NOT have exponential decay, we add $|\tilde{E}|^2$ which does have exponential decay.

$\therefore \quad \partial_t |E|^2 \leq \Delta |E|^2 - 2|\nabla E|^2 + \frac{50}{3} R|E|^2$

$$\Rightarrow \partial_\tau |\tilde{E}|^2 \leq \tilde{\Delta} |\tilde{E}|^2 - \frac{2}{37} |\tilde{\nabla} \tilde{Rc}|^2 + \frac{50}{3} \tilde{R} |E|^2 - \frac{4\tilde{r}}{3} |\tilde{E}|^2$$

$$\leq \tilde{\Delta} |\tilde{E}|^2 - \frac{2}{37} |\tilde{\nabla} \tilde{Rc}|^2 + Ce^{-\delta\tau}$$

where we used $\tilde{R} \leq C$ and $|\tilde{E}|^2 \leq Ce^{-\delta\tau}$.

Now, we define $V = |\tilde{\nabla}^k \tilde{Rc}|^2 + \alpha_{k0} |\tilde{E}|^2 + \sum_{j=1}^{k-1} \alpha_{kj} |\tilde{\nabla}^j \tilde{Rc}|^2$

w/ α_{kj}'s to be constants which we choose later. One can check that V satisfies the following evolution eq^n

$$\partial_\tau V \leq \tilde{\Delta} V + Ce^{-\delta\tau} + (B_k - 2\alpha_{k,k-1}) |\tilde{\nabla}^k \tilde{Rc}|^2$$
$$+ \sum_{j=1}^{k-1} (\alpha_{kj} B_j - 2\alpha_{k,j-1}) |\tilde{\nabla}^j \tilde{Rc}|^2$$
$$+ \left(\alpha_{k1} B_1 - \frac{2}{37} \alpha_{k0}\right) |\tilde{\nabla} \tilde{Rc}|^2 \qquad \text{--- } (1)$$

so choose α_{kj} s.t.

$$B_k - 2\alpha_{k,k-1} \leq -1$$

$$\alpha_{kj} B_j - 2\alpha_{k,j-1} \leq -\alpha_{kj} \quad \text{for } 2 \leq j \leq k-1$$

$$\alpha_{k1} B_1 - \frac{2}{37} \alpha_{k0} \leq -\alpha_{k1}$$

So, we get

$$\partial_\tau V \leq \tilde{\Delta} V + Ce^{-\delta\tau} - V$$

Now $V(0) \leq C'$ by the compactness of M and the associated ODE is

$$\frac{d\varphi}{dt} = Ce^{-\delta\tau} - \varphi$$

which has $\varphi(\tau) = Be^{-\tau} + \dfrac{C}{1-\delta} e^{-\delta\tau}$.

$\therefore$ by the maximum principle, $|\tilde{\nabla}^k \widetilde{Rc}|^2 \leq V \leq C_k e^{-\delta_k \tau}$

$\square$

Now we can do the same thing as the maximal existence time criterion to have bounds on $\partial^k \hat{\Gamma}$ and get $\tilde{g}(\tau) \to \tilde{g}_\infty$ as $\tau \to \infty$ in every C^k-norm and exponential.

$\square$